

Lack of Sphere Packing of Graphs via Non-Linear Potential Theory

Itai Benjamini Oded Schramm *

November 16, 2010

Abstract

It is shown that there is no quasi-sphere packing of the lattice grid \mathbb{Z}^{d+1} or a co-compact hyperbolic lattice of \mathcal{H}^{d+1} or the 3-regular tree $\times \mathbb{Z}$, in \mathbb{R}^d , for all d . A similar result is proved for some other graphs too. Rather than using a direct geometrical approach, the main tools we are using are from non-linear potential theory.

1 Introduction

In this paper we would like to study sphere (and quasi-sphere) packing of graphs in Euclidean spaces. Suppose that $P = (P_v : v \in V)$ is an indexed packing of spheres. This just means that V is some set, to each v corresponds a sphere $P_v \subset \mathbb{R}^n$, and the interiors of the spheres are disjoint. Let G be the graph with vertices V such that there is an edge joining v and u iff P_v and P_u are tangent. Then G is the *contacts graph* of P . (There are no multiple edges in G). In a quasi-sphere packing, we allow P_v to be any domain for which, the ratio between the outer radius and the inner radius, are uniformly bounded, for all $v \in V$. The two dimensional case is well understood, if G admits a circle packing, then it is easy to see that G is planar, the circle packing theorem provides a converse:

Theorem 1. (*Circle Packing Theorem*) *Let $G = (V, E)$ be a finite planar graph with no loops or multiple edges, then there is a disk packing $P = (P_v : v \in V)$ in \mathbb{R}^2 with contacts graph G .*

This theorem was first proved by Koebe [18]. Recently, at least 7 other proofs have been found; some of the more accessible ones can be found in [21], [7], [6].

Which graphs admit a quasi-sphere packing in $\mathbb{R}^d, d > 2$? We know of two kinds of invariants which imply that the graph cannot be packed in \mathbb{R}^d . One is the separation function of the graph. This invariant is geometrical, and in particular it can be used to show that, the lattice \mathbb{Z}^{d+1} can not be quasi-sphere packed in \mathbb{R}^d . However this invariant is not good enough to show, for instance, that a co-compact lattice in the $d+1$ dimensional hyperbolic space \mathcal{H}^{d+1} , or the 3-regular tree $\times \mathbb{Z}$ cannot be packed in \mathbb{R}^d . Note that the 3-regular tree $\times \mathbb{Z}$

*Oded died while solo climbing Guye Peak in Washington State on September 1, 2008.

admits square root separators, similar to planar graphs, although is not planar, see [2]. We will not discuss separation here, regarding many properties of the separation function for graphs, including the application above, see [4].

In this note we will use another invariant, which is formulated in the language of non-linear potential theory. In particular it gives,

Theorem 2. *The lattice \mathbb{Z}^{d+1} as well as any co-compact lattice in \mathcal{H}^{d+1} or 3-regular tree $\times \mathbb{Z}$ can not be quasi-sphere packed in \mathbb{R}^d .*

This theorem is a corollary of our main result, which is more technical.

Theorem 3. *Assume G admits a uniform bound on its vertex degree. If G is d -nonparabolic, yet admits no non constant, d -Dirichlet, d -harmonic functions, then G has no quasi-sphere packing in \mathbb{R}^d .*

All the notions in theorem 3 are defined in the next section. The rest of the sections contains the proof of this theorem, concluding with the derivation of Theorem 2. We end with a couple of problems.

Remarks:

- 1) Existence of quasi-sphere packing is a rough-isometry invariant of the graph.
- 2) Kuperberg and Schramm [19] gave bounds on the possible average kissing numbers of sphere packing in \mathbb{R}^3 . See [8] for more on fixed radius sphere packing.
- 3) $\mathcal{H}^2 \times \mathbb{R}$ is a bounded geometry Riemannian metric on \mathbb{R}^3 . For the reasons 3-regular tree $\times \mathbb{Z}$ is not sphere packable, it is possible to show that there is no packable graph in Euclidean \mathbb{R}^d roughly isometric to $\mathcal{H}^2 \times \mathbb{R}$. Which bounded geometry Riemannian metrics on \mathbb{R}^3 admit a roughly isometric packable graph?

2 Notations and Terminology

Let $G = (V, E)$ be a graph. For convenience, we usually only consider graphs with no loops or multiple edges (but the results do apply to multigraphs). We sometimes use $\{v, u\}$ to denote the edge with endpoints v, u .

The graphs we shall consider will be connected and locally finite. The latter means that the number of edges incident with any particular vertex is finite.

We now bring some useful definitions from potential theory.

Definition. The p -Dirichlet energy of a function $f : V \rightarrow \mathbb{R}$ is defined by

$$D_p(f) = 1/2 \sum_{\{v, u\} \in E} |f(u) - f(v)|^p$$

A p -Dirichlet function is an $f : V \rightarrow \mathbb{R}$ with $D_p(f) < \infty$. The space of all p -Dirichlet functions on G is denoted by $D_p(G)$.

Definition. A graph G is p -parabolic if

$$\inf\{D_p(f) \mid f \text{ has finite support, } f(v_0) = 1\} = 0 \text{ for some vertex } v_0$$

Otherwise we say that the graph is *p-nonparabolic*. Nonparabolic is called sometimes hyperbolic. 2-parabolic has a probabilistic meaning, simple random walk on the graph is recurrent. Soardi and Yamasaki [27] proved assuming a uniform bound on the degrees that the parabolic index ($= \sup\{p \geq 1 \mid G \text{ is } p\text{-parabolic}\}$), is invariant under rough isometries, (see also [25] and [17]). Maeda [20] proved that the parabolic index of \mathbb{Z}^d is exactly d , (see also [27] and [13]). It is not hard to see that the binary tree is *p-nonparabolic* for any $p \geq 1$.

For $f : V \rightarrow \mathbb{R}$, and $1 < p < \infty$, set

$$\Delta_p f(v) = \sum_{\{u \mid \{u,v\} \in E\}} |f(u) - f(v)|^{p-2} (f(u) - f(v)).$$

Definition. A function $f : V \rightarrow \mathbb{R}$ is called *p-harmonic* if $\Delta_p f(v) = 0$ at every $v \in V$.

A *metric* m on a graph $G = (V, E)$ is a positive function $m : E \rightarrow (0, \infty)$. The *gradient* of a function $f : V \rightarrow \mathbb{R}$ with respect to a metric m is defined by

$$|df(e)/m(e)| = |(f(v) - f(u))/m(e)|,$$

where $\{v, u\} = e$.

The *natural metric* on G is the metric where each edge gets weight 1. In the absence of another metric, all metric related notions are assumed to be with respect to the natural metric.

Two metrics m, m' are *mutually bilipschitz*, if the ratios m/m' and m'/m are bounded.

Let $G = (V, E)$ be a connected, locally finite graph, and let m be a metric on G . The m -length of a path γ in G is the sum of $m(e)$ over all edges in γ ,

$$\text{length}_m(\gamma) = \sum_{e \in \gamma} m(e).$$

We define the m -distance $d_m(v, u)$ between any two vertices $v, u \in V$ to be the infimum of the m -lengths of paths connecting v and u . Then (V, d_m) is a metric space.

3 *p-resolving metrics*

Below, we introduce the notion of a *p-resolvable* graph, and will see that a *p-nonparabolic p-resolvable* graph has non-constant, *p-harmonic* functions.

Let $G = (V, E)$ be some graph, and let Γ be a collection of (infinite) paths in G . Then Γ is *p-null* if there is an $L^p(E)$ metric on G such that $\text{length}_m(\gamma) = \infty$ for every $\gamma \in \Gamma$.

It is easy to see that Γ is *p-null* iff its *p-extremal length*

$$\text{EL}(\Gamma) = \sup_m \inf_{\gamma \in \Gamma} \frac{\text{length}_m(\gamma)^p}{\|m\|^p},$$

is infinite. (Extremal length was imported to the discrete setting by Duffin [9]. See [26] for more about extremal length on graphs.) When Γ is a collection of

paths and a property holds for every $\gamma \in \Gamma$, except for a p -null family, we shall say that the property holds for p -almost every $\gamma \in \Gamma$.

Let m be a metric on G , and recall that d_m is the associated distance function. Let $C_m(G)$ denote the completion of (V, d_m) , and let the m -boundary of G be $\partial_m G = C_m(G) - V$. We use d_m to also denote the metric of the completion $C_m(G)$.

The metric m will be called p -resolving if it is in $L^p(E)$ and for every $x \in \partial_m G$ the collection of half infinite paths in G that converge to x in $C_m(G)$ is p -null. G is p -resolvable if it has a p -resolving metric.

Note that if m is a p -resolving metric and m' is another L^p metric satisfying $m' \geq m$, then m' is also p -resolving.

Theorem 4 below shows that any p -parabolic graph is p -resolvable, in fact, there is a metric m with $\partial_m G = \emptyset$. On the other hand, the next theorem shows that a p -nonparabolic graph with no non-constant, p -harmonic functions, is not p -resolvable, for example, \mathbb{Z}^3 or any co-compact lattice in hyperbolic d -space $d > 2$ is not 2-resolvable. Any bounded valence planar graph is 2-resolvable [2].

Theorem 4. *Let $G = (V, E)$ be an infinite, connected, locally finite, p -resolvable graph. If G is p -nonparabolic, then there are non-constant, p -Dirichlet p -harmonic functions on G .*

We shall need the following results.

Theorem 5. (Yamasaki [28]) *Let G be a locally finite connected graph, and let Γ be the collection of all infinite paths in G . Then G is p -parabolic if and only if Γ is p -null.*

(Yamasaki [28] considers only paths that start at a fixed base vertex, but this is equivalent.)

Theorem 6. (Yamasaki [29]) *There are non-constant, p -Dirichlet, p -harmonic, functions on G if and only if there is an $f \in D^p(G)$ such that for every $c \in \mathbb{R}$ the collection of all one-sided-infinite paths γ in G with $\lim_n f(\gamma(n)) \neq c$ is not p -null.*

Proof of theorem 4: Assume that G is p -nonparabolic, and m is a p -resolving metric on G . Let Γ be the collection of all infinite paths $\gamma = (\gamma(0), \gamma(1), \dots)$ in G . Almost all paths γ in Γ have a limit $\lim_n \gamma(n)$ in $\partial_m G$, since the m -length of those that do not is infinite. (The limit is in the metric d_m , of course.)

We now define $\text{supp}(\Gamma)$, the *support* of Γ in $\partial_m G$, as the intersection of all closed sets $Q \subset \partial_m G$ such that for almost every $\gamma \in \Gamma$ the limit $\lim_n \gamma(n)$ is in Q . Because there is a countable basis for the topology of $\partial_m G$, and a countable union of p -null collections of curves is p -null, almost every $\gamma \in \Gamma$ satisfies $\lim_n \gamma(n) \in \text{supp}(\Gamma)$.

Since G is p -nonparabolic, we know from theorem 5 that the extremal length of Γ is finite. Consequently, $\text{supp}(\Gamma)$ is not empty. Moreover, the assumption that m is p -resolving shows that $\text{supp}(\Gamma)$ consists of more than a single point. Let x_0 be an arbitrary point in $\text{supp}(\Gamma)$. Define $f : V \rightarrow \mathbb{R}$ by setting $f(p) = d_m(x_0, p)$. It is clear that $|df(e)| \leq m(e)$ holds for $e \in E$. Consequently, $f \in D^p(G)$.

Pick some $\delta > 0$ that is smaller than the d_m -diameter of $\text{supp}(\Gamma)$. Consider the set $A_\delta = \{x \in \text{supp}(\Gamma) : d(x_0, x) < \delta\}$, and let Γ_δ be the set of $\gamma \in \Gamma$ such

that $\lim_n \gamma(n) \in A_\delta$. Since $\text{supp}(\Gamma)$ is not contained in \bar{A}_δ or in $\partial_m G - A_\delta$, from the definition of $\text{supp}(\Gamma)$ it follows that both Γ_δ and $\Gamma - \Gamma_\delta$ are not p -null. For every $\gamma \in \Gamma_\delta$, we have $\lim_n f(\gamma(n)) < \delta$, while for almost every $\gamma \in \Gamma - \Gamma_\delta$, we have $\lim_n f(\gamma(n)) \geq \delta$. Since both Γ_δ and $\Gamma - \Gamma_\delta$ are non p -null, it follows that for every constant $c \in \mathbb{R}$ the set of $\gamma \in \Gamma$ with $\lim_n f(\gamma(n)) \neq c$ is not p -null.

Now theorem 6 implies that there are non-constant, p -harmonic functions, on G . \square

4 Sphere packing in \mathbb{R}^d and a d -resolving metric

In this section we will show that a quasi-sphere packing in \mathbb{R}^d provides a d -resolving metric for G . The argument is similar to the case $d = 2$ in [2]. This will allow us to finish the proof of Theorem 3.

Theorem 7. *Let G be a graph with uniformly bounded vertex degree. If G admits a quasi-sphere packing in \mathbb{R}^d , then G admits a d -resolving metric.*

Proof: Assume there is a quasi-sphere packing of G in \mathbb{R}^d , thus it admits a packing in S^d . Take any $e \in E$, and let its vertices be u, v . We set $m(e) = \text{diam}(P_u) + \text{diam}(P_v)$. This defines a metric $m : E \rightarrow (0, \infty)$. Because the packing P is contained in S^d , its total volume is finite, and this implies that $m \in L^d(E)$.

We shall now show that m is d -resolving. For any $v \in V$ we let $z(v)$ denote the center of the disk P_v . Let p be any point in $\partial_m G$. Let v_1, v_2, \dots be a sequence in V that converges to p in the completion of G , with respect to m , $C_m(G)$. Then $\lim_{n,k \rightarrow \infty} d_m(v_n, v_k) = 0$. This easily implies that $\lim_{n,k \rightarrow \infty} |z(v_n) - z(v_k)| = 0$. We therefore conclude that the limit $\lim_n z(v_n)$ exists, and denote this limit by $z(p)$. If w_1, w_2, \dots is another sequence in V that converges to p , then the limit $\lim_n z(w_n)$ will still be $z(p)$. This follows from the fact that any ordering of the union $\{v_n\} \cup \{w_n\}$ as a sequence will still converge to p . Hence $z(p)$ does not depend on the sequence chosen.

Let Γ_p be the collection of all half-infinite paths $\gamma = (\gamma(0), \gamma(1), \dots)$ in G that converge to p in $C_m(G)$. We need to show that Γ_p is d -null. This will be done by producing an $L^d(E)$ metric m_p such that $\text{length}_{m_p}(\gamma) = \infty$ for every $\gamma \in \Gamma_p$. The argument will be similar to an argument in [24] and [12].

We now inductively construct a sequence of positive numbers $r_1 > r_2 > r_3 > \dots$. For $r > 0$, let $B(r)$ denote the disk $\{z \in S^d : |z - z(p)| < r\}$. Take $r_1 = 1$. Suppose that r_1, r_2, \dots, r_{n-1} have been chosen. Let r_n be in the range $0 < r_n < r_{n-1}/2$ and be sufficiently small so that the two sets of vertices $\{v \in V : z(v) \in B(2r_n)\}$ and $\{v \in V : z(v) \notin B(r_{n-1})\}$ are disjoint and there is no edge in G connecting them. To see that this can be done, observe that for any $r > 0$ there are finitely many vertices $v \in V$ such that $\text{diam}(P_v) \geq r$. Since there is a uniform bound on the degrees, for every $r > 0$ there is a $\rho(r) \in (0, r/2]$ such that the closure of $B(\rho(r))$ does not intersect any of the sets P_v satisfying $\text{diam}(P_v) \geq r/2$. This implies that there will be no P_v that intersects both circles $\partial B(r)$ and $\partial B(\rho(r))$. Hence we may take $r_n = \rho(r_{n-1})/2$.

For $r \in (0, \infty)$, let $\psi_r : \mathbb{R}^d \rightarrow [0, r]$ be defined by

$$\psi_r(z) = \begin{cases} r & \text{if } |z - z(p)| \leq r, \\ 2r - |z - z(p)| & \text{if } r \leq |z - z(p)| \leq 2r, \\ 0 & \text{if } |z - z(p)| \geq 2r. \end{cases}$$

In other words, ψ_r is equal to r on $B(r)$, equal to 0 outside $B(2r)$, and in the annulus $B(2r) - B(r)$ it is linear in the distance from its center $z(p)$. For each $n = 1, 2, \dots$ and $v \in V$, we define

$$\phi_n(v) = \psi_{r_n}(z(v)).$$

The construction of the sequence r_1, r_2, \dots insures that the supports of $d\phi_n$ and $d\phi_{n'}$ are disjoint when $n \neq n'$. It is easy to see that the definition of ϕ_n shows that there is a finite constant C such that

$$|d\phi_n(e)|^d \leq C \text{vol}((P_u \cup P_v) \cap B(3r_n)), \quad (1)$$

where $\{u, v\} = e$. Let Ω be an upper bound on the degrees of the vertices in G . Since the interiors of the sets in P are disjoint, (1) implies

$$\|d\phi_n\|_d^d = \sum_{e \in E} |d\phi_n(e)|^d \leq 3^d \omega_d \pi C \Omega r_n^d,$$

where $\omega_d r^d$ is an upper bound for the volume of an r -ball in S^d .

Now set

$$\phi = \sum_{n=1}^{\infty} \frac{\phi_n}{nr_n}.$$

As we have noted, the supports of the different $d\phi_n$ are disjoint, and therefore,

$$\|d\phi\|_d^d = \sum_{n=1}^{\infty} \frac{\|d\phi_n\|_d^d}{n^d r_n^d},$$

and the above estimate for $\|d\phi_n\|_d^d$ shows that $|d\phi| \in L^d(E)$. Therefore, there is some metric $m_p \in L^d(E)$ such that $m_p(e) \geq |\phi_n(e)|$ for every $e \in E$. (Technically, we cannot take $m_p = |d\phi|$, since $|d\phi|$ is not positive, and hence not a metric.)

Now let $\gamma = (\gamma(1), \gamma(2), \dots)$ be any path in Γ_p , and let $E(\gamma)$ denote its edges. We have $\lim_n z(\gamma(n)) = z(p)$. Therefore,

$$\lim_n \phi(\gamma(n)) = \lim_{z \rightarrow z(p)} \sum_j \frac{\psi_{r_j}(z)}{jr_j} = \sum_j \frac{1}{j} = \infty.$$

This gives

$$\text{length}_{m_p}(\gamma) = \sum_{e \in E(\gamma)} m_p(e) \geq \sum_{e \in E(\gamma)} |d\phi(e)| = \infty.$$

So Γ_p is d -null, and m is d -resolving. □

Proof of Theorem 3: Follows from Theorems 4 and 7. □

Proof of Theorem 2: By Theorem 3 it is enough to note that \mathbb{Z}^{d+1} , co-compact lattices in \mathcal{H}^{d+1} and 3-regular tree $\times \mathbb{Z}$ are d -nonparabolic, yet admit no non constant d -harmonic functions.

For \mathbb{Z}^{d+1} , the first part can be found in [20] (see also [13]), the second in [16].

Hyperbolic space \mathcal{H}^d has strictly positive Cheeger constant. This property passes to co-compact lattices, see chapter 6 in [10]. Let $G = (V, E)$ be the Cayley graph of a co-compact lattice in \mathcal{H}^d . By Cheeger's inequality,

$$\sum_{v \in V} |f(v)| \leq c \sum_{e \in E} |df(e)|,$$

for f with compact support. Applying this to $f = u^p$, $p \geq 1$, yields

$$\sum_{v \in V} |f(v)|^p \leq c \sum_{e \in E} |df(e)|^p.$$

This, in turn, implies that G is not p -parabolic, see also [15]. The same argument applies to 3-regular tree $\times \mathbb{Z}$.

To deal with p -Dirichlet p -harmonic functions, we need some preliminaries.

If $f \in \ell^p(V)$ and $g \in \ell^{p'}(V)$, $p' = \frac{p}{p-1}$, let us denote by

$$\langle f, g \rangle = \sum_{v \in V} f(v)g(v).$$

The p -Laplacian is related to the gradient of D_p in the following way. If $\Delta_p f \in \ell^{p'}(V)$ and $g \in \ell^p(V)$,

$$\frac{\partial}{\partial t} D_p(f + tg)_{t=0} = -p \langle g, \Delta_p f \rangle.$$

Since D_p is homogeneous of degree p , differentiating $D_p(f + tf) = (1+t)^p D_p(f)$ leads to the following identity,

$$\langle f, \Delta_p f \rangle = -D_p(f),$$

for $f \in \ell^p(V)$ and $\Delta_p f \in \ell^{p'}(V)$. In particular, a non-constant p -Dirichlet p -harmonic function cannot be L^p -integrable. Thus it suffices to show that for the graphs of interest, every p -Dirichlet function is L^p -integrable, up to a constant. This suggests the following definition. The degree 1 exact L^p -cohomology (L^p -cohomology for short) of a Riemannian manifold or a graph is the quotient of the space of p -Dirichlet functions by the subspace generated by constants and L^p -integrable p -Dirichlet functions.

We have just shown that constants are the only p -harmonic functions in the trivial cohomology class. More generally, if $p > 1$, a cohomology class contains at most one p -harmonic function, up to constants. Indeed, since D_p is strictly convex, if $g \neq 0$, $g \in \ell^p(V)$ and $\Delta_p f \in \ell^{p'}(V)$,

$$D_p(f + g) > D_p(f) - p \langle g, \Delta_p f \rangle.$$

So if f is p -harmonic, $D_p(f + g) > D_p(f)$. If $f + g = h$ is p -harmonic as well, $D_p(f) = D_p(h - g) > D_p(h)$, contradiction.

The translation by one in the \mathbb{Z} factor is an isometry τ of the graph $G = 3$ -regular tree $\times \mathbb{Z}$. If f is a function on vertices of G , and if $D_p(f) < \infty$, then $f \circ \tau - f$ is L^p -integrable, i.e. f and $f \circ \tau$ belong to the same L^p -cohomology

class. If f is p -harmonic, so is $f \circ \tau$. This implies that $f \circ \tau = f$, i.e. f is \mathbb{Z} -invariant. Since $D_p(f) < \infty$, this can hold only if f is constant. So p -Dirichlet p -harmonic functions on G are constant, for all $p > 1$.

We use the rough isometry invariance of L^p -cohomology, not formally stated but proven in [22], formally stated in chapter 8 of [11], see also [5]: if a bounded degree graph G is roughly isometric to a bounded geometry Riemannian manifold M , then, for all $p \geq 1$, the L^p -cohomologies of G and M are isomorphic. A result of [22] states that L^p -cohomology of \mathcal{H}^{d+1} vanishes if and only if $p \leq d$. It follows that co-compact lattices in hyperbolic space \mathcal{H}^{d+1} have no non constant d -Dirichlet d -harmonic functions. \square

Remark: In the course of the proof, we have shown that if a graph admits non constant p -Dirichlet p -harmonic functions, then its L^p -cohomology does not vanish. This yields the following corollary of Theorem 3.

Corollary 8. *If a bounded degree, d -nonparabolic graph can be quasi-sphere packed into \mathbb{R}^d , then its L^d -cohomology does not vanish.*

Here is a consequence of Corollary 8 in the world of Cayley graphs of co-compact lattices in connected Lie groups.

In [23], it is shown that if a connected Lie group has non vanishing L^p -cohomology for some p , then it is roughly isometric either to a negatively curved Lie group, or to a compact by solvable unimodular Lie group. Thus a co-compact lattice in a connected Lie group which has non vanishing L^p -cohomology for some p is either Gromov hyperbolic or amenable. In particular, lattices in higher rank semi-simple Lie groups cannot be quasi-sphere packed into \mathbb{R}^d for any d .

5 Further problems

Remark: Holopainen and Soardi [16] proved the Liouville theorem for p -harmonic functions on any G , for which a weak Poincaré inequality holds and has the volume doubling property. This can be used to show that the Cayley graph of any group of polynomial volume growth with power $\geq d + 1$, cannot be (quasi-) sphere packed in \mathbb{R}^d .

Problem 1. *Can Cayley graphs of polynomial volume growth groups be packed in \mathbb{R}^d , for some d ?*

Problem 2. *Show that any sphere packing of \mathbb{Z}^3 in S^3 , has only one accumulation point for the centers of the spheres.*

Problem 3. *By compactness it follows, that already a sufficiently large ball in \mathbb{Z}^{d+1} , cannot be packed in \mathbb{R}^d . Estimate the size of such a ball.*

Problem 4. *Characterize or give necessary and sufficient conditions regarding which bounded geometry Riemannian metrics on \mathbb{R}^3 admit a rough isometric graph, packable in \mathbb{R}^3 .*

Problem 5. *Packing in other spaces. Separation can be used to show that \mathbb{Z}^d can not be packed in \mathcal{H}^d . What about packing of \mathbb{Z}^3 or co compact lattices of \mathcal{H}^3 in $\mathcal{H}^2 \times \mathbb{R}$?*

The recent paper [1] uses Theorem 7 above in order to extend parts of [3] to higher dimensions and suggests further related problems as well.

Acknowledgements. Thanks to I. Holopainen and L. Saloff-Coste for a useful discussion. Special thanks to P. Pansu for writing the last part using L^p -cohomology.

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